

§ 16.7 General coordinate transformations —

- We used the geometry of polar/cylindrical/spherical coordinate transformations to get the amplification factor for area/volume

Polar: $dx dy = r dr d\theta$

Cylindrical: $dx dy dz = r dr d\theta dz$

Spherical: $dx dy dz = r^2 \sin\theta dr d\theta d\phi$

- How does it work in general?

2-D

$$x = g(u, v)$$

$$y = h(u, v)$$

or

$$x = g(u, v, w)$$

$$y = h(u, v, w)$$

$$z = k(u, v, w)$$

- Compare with the 1-dimensional 2
Substitution principle of Math 21B

$$\int_a^b f(x) dx = \int_{u_a}^{u_b} f(g(u)) \underbrace{g'(u)}_A du$$

$u_b \quad b = g(u_b)$
 $u_a \quad a = g(u_a)$

$$x = g(u)$$

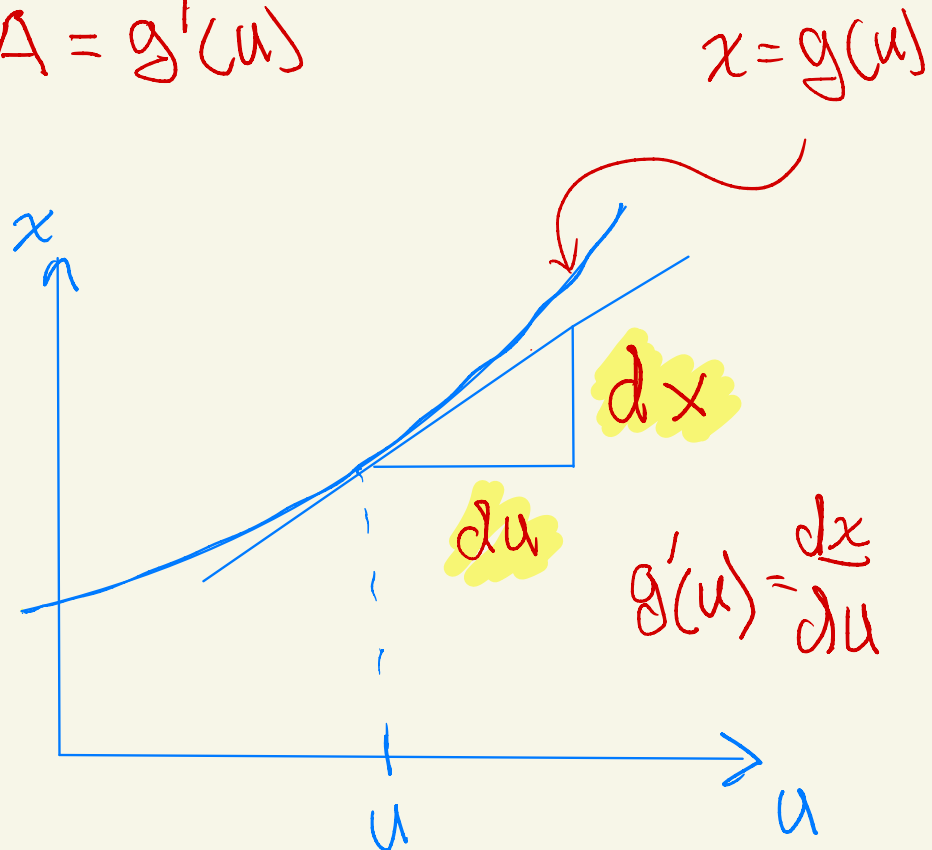
$$dx = \boxed{g'(u)} du$$

Amplification
factor for length
 $A = g'(u)$

Picture:

$$m = g'(u) = \frac{dx}{du}$$

$$\underbrace{dx}_A = g'(u) du$$



- Consider 2-dimensions

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$$I = \iint_{R_{xy}} f(x, y) dx dy$$

$$\begin{aligned} x &= g(u, v) \\ y &= h(u, v) \end{aligned}$$

Theorem :

$$\iint_{R_{xy}} f(x, y) dx dy$$

$$= \iint_{R_{uv}} f(g(u, v), h(u, v)) A du dv$$

$$\begin{aligned} A &= \det \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix} = \det \begin{vmatrix} -\nabla g \\ -\nabla h \end{vmatrix} \\ &= g_u h_v - g_v h_u \end{aligned}$$

Ex Check Polar Coordinates:

(4)

$$x = r \cos \theta = g(r, \theta) \quad u = r$$
$$y = r \sin \theta = h(r, \theta) \quad v = \theta$$

$$\nabla g = \left(\frac{\partial g}{\partial r}, \frac{\partial g}{\partial \theta} \right) = (\cos \theta, -r \sin \theta)$$

$$\nabla h = \left(\frac{\partial h}{\partial r}, \frac{\partial h}{\partial \theta} \right) = (\sin \theta, r \cos \theta)$$

$$A = \det \begin{vmatrix} -\nabla g \\ -\nabla h \end{vmatrix} = \det \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$$

$$= \overset{g_u}{(\cos \theta)} \overset{h_v}{(r \cos \theta)} - \overset{g_v}{(-r \sin \theta)} \overset{h_u}{(\sin \theta)}$$

$$= r \cos^2 \theta + r \sin^2 \theta = r(\cos^2 \theta + \sin^2 \theta)$$

$$= r \quad \checkmark$$

Summary :

(5)

$$\tilde{f}(u,v) = f(g(u,v), h(u,v))$$

$$\text{2-D: } \iint_{R_{xy}} f(x,y) \, dA_{xy} = \iint_{R_{uv}} \tilde{f}(u,v) \begin{vmatrix} -\nabla g \\ -\nabla h \end{vmatrix} du \, dv$$

$\uparrow$
 $x = g(u,v)$
 $y = h(u,v)$

$\uparrow$
 $\det \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix}$

$$\text{3-D: } \iiint_{D_{xyz}} f(x,y,z) \, dA_{xyz} = \iiint_{D_{uvw}} \tilde{f}(u,v,w) \begin{vmatrix} -\nabla g \\ -\nabla h \\ -\nabla k \end{vmatrix} du \, dv \, dw$$

$\uparrow$
 $x = g(u,v,w)$
 $y = h(u,v,w)$
 $z = k(u,v,w)$

$\uparrow$
 $\det \begin{vmatrix} g_u & g_v & g_w \\ h_u & h_v & h_w \\ k_u & k_v & k_w \end{vmatrix}$

⑥
• Notation: Since the Amplification factor is given by a Jacobian Determinant, we usually denote it by J

Eg: $x = g(u, v)$
 $y = h(u, v)$

straight lines
mean determinant

$$J = \det \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix} \equiv \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix}$$

"use J instead of A "

Sometimes — $J = \begin{pmatrix} g_u & g_v \\ h_u & h_v \end{pmatrix} \otimes$

$$|J| = \det \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix}$$

• More Notation -

(7)

When we write

$$J = \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix} \begin{matrix} \leftarrow \text{1st row } \nabla g \\ \leftarrow \text{2nd row } \nabla h \end{matrix}$$

$$\nabla g = (g_u, g_v) = \left(\frac{\partial g}{\partial u}(u, v), \frac{\partial g}{\partial v}(u, v) \right)$$

$$\nabla h = (h_u, h_v) = \left(\frac{\partial h}{\partial u}(u, v), \frac{\partial h}{\partial v}(u, v) \right)$$

Recall:

$$\frac{\partial g}{\partial u}(u, v) = \lim_{\Delta u \rightarrow 0} \frac{g(u + \Delta u, v) - g(u, v)}{\Delta u}$$

"slope in direction u at pt (u, v) "

$\nabla g(u, v)$ points in direction of steepest increase

This: a coordinate change is given by:

(8)

$$(u, v) \longrightarrow (x, y)$$

$$x = g(u, v)$$

$$y = h(u, v)$$

$$J = \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix} = \begin{vmatrix} -\nabla g \\ -\nabla h \end{vmatrix} \begin{matrix} \leftarrow \text{1st row } \nabla g \\ \leftarrow \text{2nd row } \nabla h \end{matrix}$$

Another notation:

$$J = \left| \frac{\partial(x, y)}{\partial(u, v)} \right|,$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

I.e. $x = g(u, v)$

$$\frac{\partial x}{\partial u} = g_u = \frac{\partial g}{\partial u}(u, v)$$

Matrix
of
Partial
Derivatives

Example 5

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① Assume

$$x = 2u + 3v$$

$$y = -u + v$$

Find J such that $dx dy = J du dv$

Soln: $J = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$$= \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = 2 + 3 = 5$$

$$dx dy = 5 du dv.$$

So:

$$\iint_{R_{xy}} f(x, y) dV_{xy} = \iint_{R_{uv}} f(2u+3v, -u+v) 5 dV_{uv}$$

(2) Assume:

$$x = 2u - v - w$$

$$y = u + v$$

$$z = u + v + w$$

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Find J st $dx dy dz = J du dv dw$

$$\text{Soln: } J = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = \begin{vmatrix} 2 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

"To evaluate, expand by any row/col"

2nd row:

$$J = (-1)^{2+1} \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} + (-1)^{2+2} \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + (-1)^{2+3} \cdot 0 \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= (-1)(-1+1) + 1(2+1) = 3$$

$$dx dy dz = 3 du dv dw$$

(Continued)

(11)

(2) Assume:

$$x = 2u - v - w$$

$$y = u + v$$

$$z = u + v + w$$

$$J = 3 \quad \text{so}$$

$$\iiint_{D_{xyz}} f(x, y, z) dV_{xyz} = \iiint_{D_{uvw}} \bar{f}(u, v, w) \underbrace{3}_{J} dV_{uvw}$$

$$\bar{f}(u, v, w) = f(2u - v - w, u + v, u + v + w)$$

Eg: If $f = 1$ and D_{xyz} is the region st D_{uvw} is $0 \leq u \leq 1, 0 \leq v \leq 2,$

$$\iiint_{D_{uvw}} dV_{xyz} = \int_0^3 \int_0^2 \int_0^1 3 du dv dw = 3 \cdot 3 \cdot 2 = 18$$

Q: Why does it work?

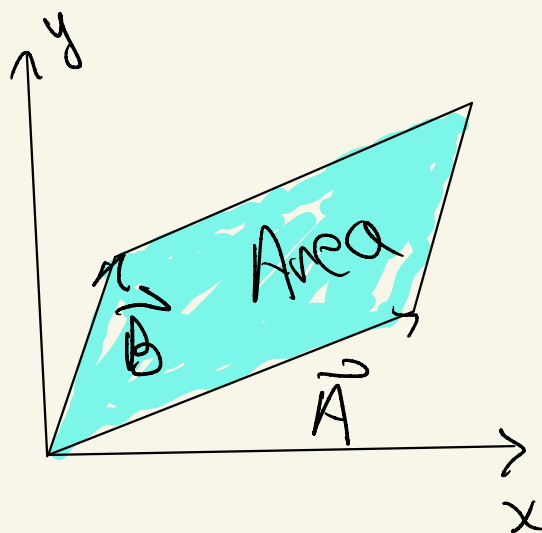
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Ans: We assume the following fundamental property of Determinants

Thm ($\mathbb{R}^2$) $\det \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \text{Area of parallelogram}$

$$\vec{A} = (a, c) \quad \vec{B} = (b, d)$$

$$\text{Area} = \det \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$



Now consider $(u, v) \rightarrow (x, y)$

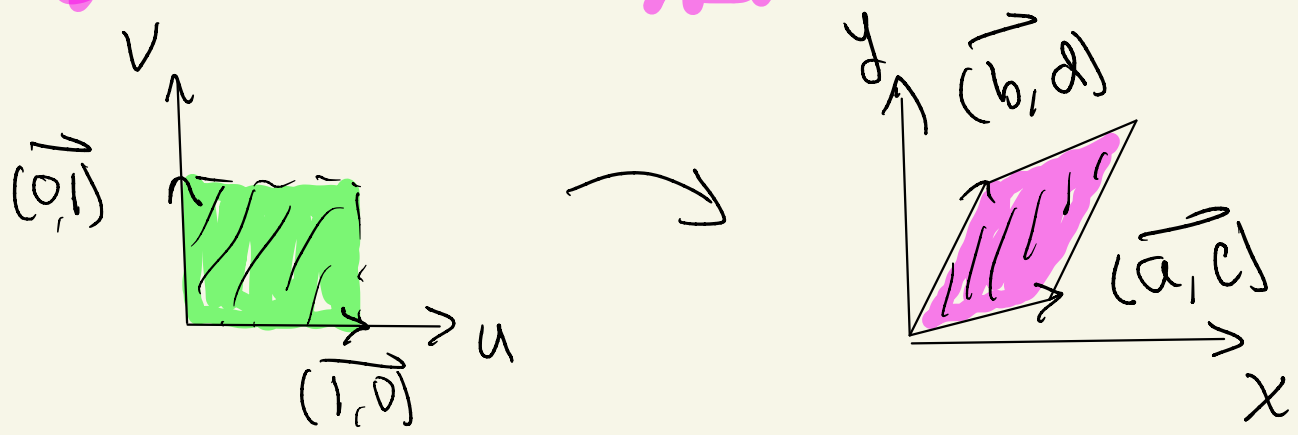
$$\begin{aligned} x &= au + bv \\ y &= cu + dv \end{aligned}$$

This is a linear transformation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

How does it map unit square $(1,0) \times (0,1)$?

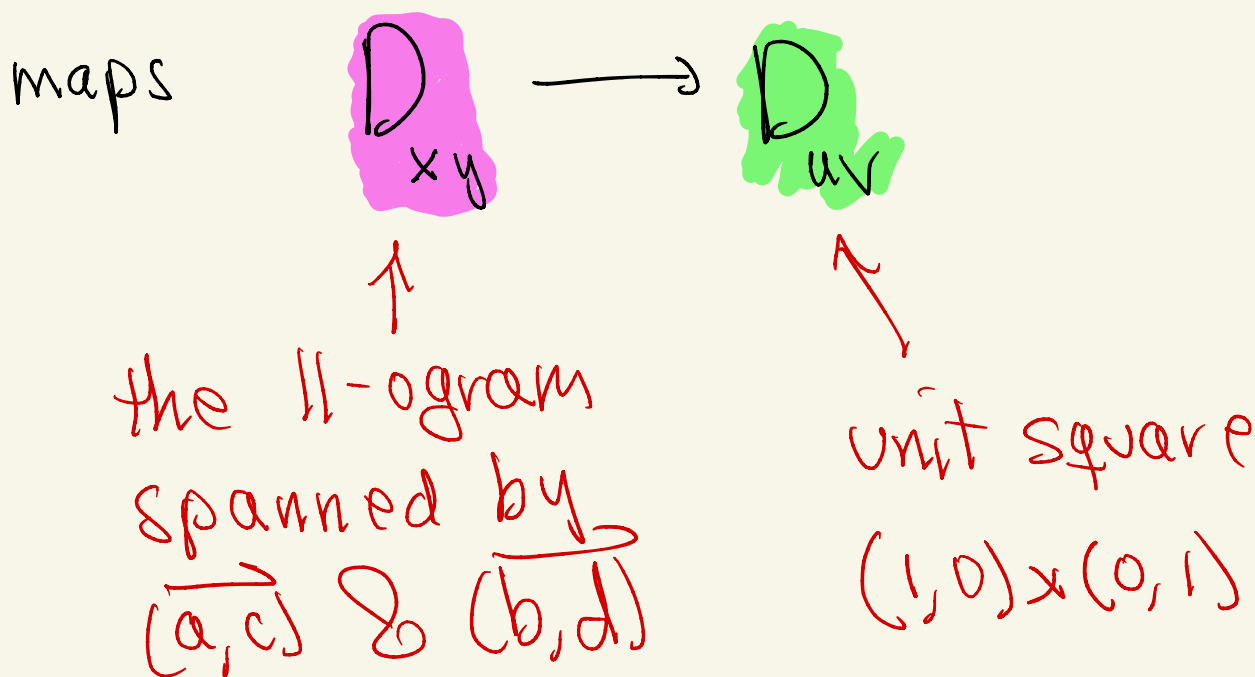
$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; \quad \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



$$(1,0) \times (0,1) \rightarrow (a,c) \times (b,d)$$

uv -area $\quad\quad\quad xy$ -area

Conclude: The map $(u, v) \rightarrow (x, y)$



Thus: $dx dy = J du dv$

In general: $x = g(u, v)$
 $y = h(u, v)$

at (u, v) is approximated by mapping

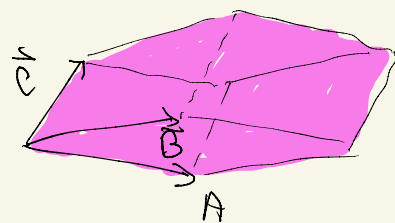
$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \underbrace{\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}}_J \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} \Rightarrow dx dy = |J| du dv$$

same reason!

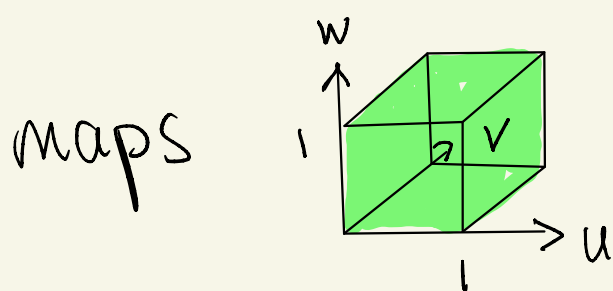
Similarly in $\mathbb{R}^3$:

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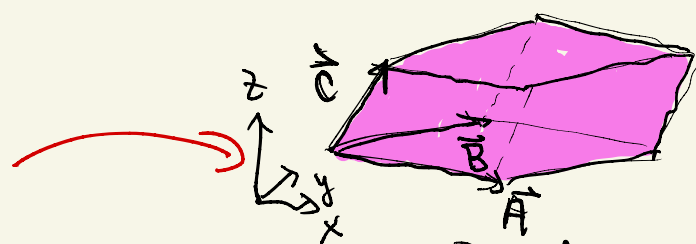
$$\det \begin{bmatrix} 1 & 1 & 1 \\ \vec{A} & \vec{B} & \vec{C} \\ 1 & 1 & 1 \end{bmatrix} = \text{Area of 1-0 piped}$$



$$\text{So } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \vec{A} & \vec{B} & \vec{C} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$



unit 3-cube
in (u, v, w)



1-0 piped
spanned by
columns of Matrix

$$\text{Conclude: } dx dy dz = J du dv dw$$

$$\text{In general: } \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \frac{\partial(x, y, z)}{\partial(u, v, w)} \end{bmatrix} \begin{bmatrix} du \\ dv \\ dw \end{bmatrix}$$

$\Rightarrow$ Same Result

Orientation: The Jacobian

keeps track of the order (u, v)

Defn: If $(u, v) \xrightarrow{\text{map}} (x, y)$ is given

by $x = g(u, v)$ and $y = h(u, v)$ and $J = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \neq 0$

Then:

$J > 0$ orientation preserving

$J < 0$ orientation reversing

Same in $\mathbb{R}^3$:

$$x = g(u, v, w)$$

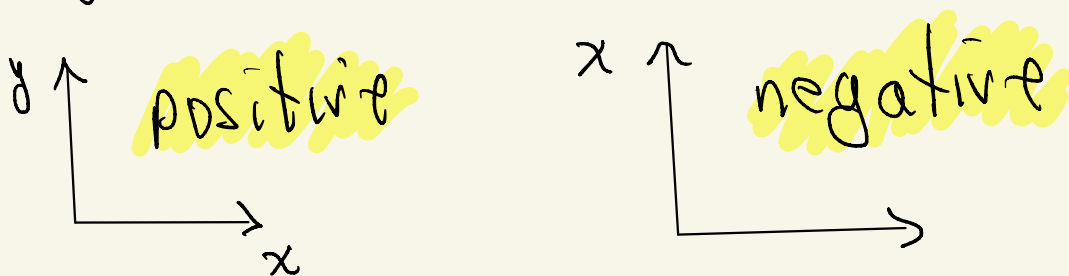
$$y = h(u, v, w)$$

$$z = k(u, v, w)$$

$$J = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right|$$

Application:

$\mathbb{R}^2$: We say a coordinate system (x, y) is positively oriented if "there's 90° betw. pos x & pos y "

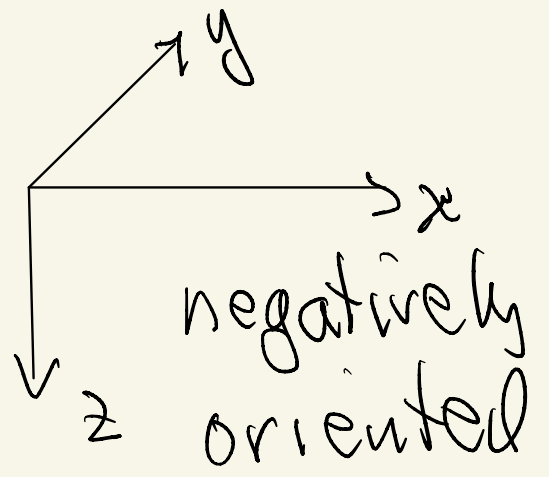
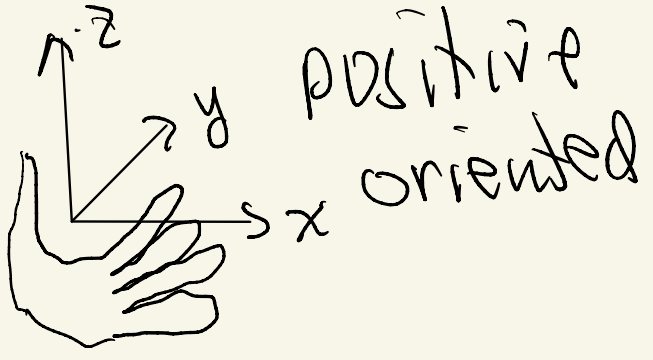


Thm: For $(u, v) \rightarrow (x, y)$, $J > 0$ iff (u, v) has same orientation as (x, y)

Similarly in $\mathbb{R}^3$

$\mathbb{R}^3: (x, y, z)$ is positively oriented if

Right Hand Rule holds



Thm: If (x, y, z) is positively oriented, then so is (u, v, w)

if and only if $J = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| > 0$.

Example: Assume (x, y) is positively oriented, and $x = r \cos \theta$
 $y = r \sin \theta$ (19)

Determine whether (r, θ) or (θ, r) is positively oriented.

We have:

$$J = \begin{vmatrix} \frac{\partial(x, y)}{\partial(r, \theta)} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r$$

$$\text{Check: } \begin{vmatrix} \frac{\partial(x, y)}{\partial(\theta, r)} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial r} \end{vmatrix} = -r$$

"Determinant changes sign if you interchange two rows or columns"

Similarly in $\mathbb{R}^3$

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$$J = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Conclude: if
 $J > 0$, then

interchange
two columns
 $\Rightarrow$ chng sign

$$\left| \frac{\partial(x, y, z)}{\partial(u, w, v)} \right| < 0, \quad \left| \frac{\partial(x, y, z)}{\partial(w, u, v)} \right| > 0, \text{ etc}$$

one
interchange

two
interchanges